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The harmonious chromatic number of complete r-ary trees

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Abstract

A *harmonious colouring* of a simple graph G is a proper vertex colouring such that each pair of colours appears together on at most one edge. The *harmonious chromatic number* $h(G)$ is the least number of colours in such a colouring. We define $Q(m)$ to be the least positive integer k such that $\binom{k}{2} \geqslant m$. Then $h(G) \geqslant Q(m)$ for any graph G with m edges. We consider the complete r-ary tree of height H, denoted $T_{r,H}$. We show that for any $r \geqslant 2$, $H \geqslant 3$, if m is the number of edges of $T_{r,H}$, then $h(T_{r,H}) = Q(m)$, except that $h(T_{2,3}) = 7$.

Keywords: Harmonious chromatic number; Tree

1. Introduction

A *harmonious colouring* of a simple graph G is a proper vertex colouring such that each pair of colours appears together on at most one edge. Formally a harmonious colouring is a function c from a colour set C to the set $V(G)$ of vertices of G such that for any edge e of G, with endpoints x, y say, $c(x) \neq c(y)$, and for any pair of distinct edges e, e', with endpoints x, y and x', y', respectively, $\{c(x), c(y)\} \neq \{c(x'), (y')\}$.

If we have a harmonious colouring of G with k colours, then since each pair of colours can appear on at most one edge, it is clear that the number of colour pairs, namely $\binom{k}{2}$, must be at least m, the number of edges of G. This motivates the following definition.

Definition. Let m be a positive integer. Then we define $Q(m)$ to be the least positive integer k such that $\binom{k}{2} \geqslant m$.

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It is easily calculated that $Q(m)=\lceil\frac{1}{2}(1+\sqrt{8m+1})\rceil$. For any graph G with m edges, $h(G)\geqslant Q(m)$.

In this paper we look at a very special class of graphs, namely the complete r-ary trees of height H. The complete r-ary tree of height H has a single root vertex (at level 0), and all nodes in levels $0,\dots,H-1$ have exactly r children, the vertices in level H being leaves. We denote this tree by $T_{r,H}$.

Several authors [5–8] have considered the problem of determining $h(T_{r,H})$. The book by Jensen and Toft [4] also lists it as Problem 17.5. It is obvious that $h(T_{r,1})=r+1$, and Mitchem [8] showed that $h(T_{r,2})=\lceil 3(r+1)/2\rceil$ for each r. Zhikang Lu [5,6] gave some nearly exact results for $r=2$ and estimates for $r=3,4$. Finally it is shown in [1] that for each r, $h(T_{r,H})=Q(m)$ provided that H is large enough.

In this paper we solve all remaining cases, and establish that $h(T_{r,H})=Q(m)$ for all $r\geqslant 2$, $H\geqslant 3$ except that $h(T_{2,3})=7=Q(m)+1$.

Note that we will always assume that m refers to the number of edges of the graph being considered, so that $Q(m)$ is the lower bound on h given above. It is easily calculated that the number of vertices of $T_{r,H}$ is

$$n=\sum_{i=0}^{H} r^i=(r^{H+1}-1)/(r-1),$$

of which r^H are leaves, and the number of edges is

$$m=n-1=\frac{r}{r-1}(r^H-1).$$

2. Orientation lemma

In this section we state and prove a lemma on orienting the edges of a graph. This lemma will be used repeatedly, to extend colourings of the non-leaf nodes of a tree to the whole tree.

First we need some notation and two lemmas from earlier papers. If S is a set of vertices of a graph G, then define P_S to be the set of edges which have at least one endpoint in the set S. The following lemma is proved in [1].

Lemma 2.1. *Let $G=(V,E)$ be a graph with n vertices cyclically arranged such that each vertex is joined to the $2k$ vertices nearest to it ($n\geqslant 2k+1$). Let $S\subseteq V$ with $k\leqslant |S|\leqslant n-k$. Then*

$$|P_S|\geqslant |S|k+k^2/4.$$

The other lemma gives a general condition for an orientation of the graph with given outdegrees to exist. It is an easy consequence of Hall's Marriage Theorem and is proved in [3].

Lemma 2.2. *Let $G=(V,E)$ be a graph, and for each vertex $v\in V$ let k_v be a non-negative integer. Then the edges of G can be oriented so that for each vertex v, the outdegree of v is at least k_v, provided that*

$$\textit{for each } S\subseteq V, \quad |P_S|\geqslant \sum_{v\in S} k_v.$$

We can now prove the main lemma.

Lemma 2.3. *Let $G=(V,E)$ be a graph with n vertices cyclically arranged such that each vertex is joined to at least $2k-t$ of the $2k$ vertices nearest to it* ($n\geqslant 2k+1$) *and the total number of edges is m. For each vertex v let k_v be a positive integer. Then the edges can be oriented so that for each vertex v, the outdegree of v is at least k_v, provided that* (i) $m\geqslant \sum_{v\in V} k_v$, *and* (ii) *for each $v\in V$, $k/2\leqslant k_v\leqslant k'$ where $k'=k-t+k^2/4(n-k)$.*

Proof. By Lemma 2.2, we need to show that for each $S\subseteq V$, $|P_S|\geqslant \sum_{v\in S} k_v$. First note that since $n\geqslant 2k+1$, we have $n-k\geqslant k+1$, so that

$$k'=k+\frac{k^2}{4(n-k)}-t\leqslant k+\frac{k^2}{4(k+1)}-t\leqslant \frac{5k}{4}-t<\frac{3k}{2}-t.$$

There are three cases:

1. $k\leqslant |S|\leqslant n-k$. Then by Lemma 2.1 we have

$$\begin{aligned}|P_S| &\geqslant (k-t)|S|+\frac{k^2}{4}\geqslant (k-t)|S|+\frac{k^2}{4}\left(\frac{|S|}{n-k}\right)\\ &=\left(k-t+\frac{k^2}{4(n-k)}\right)|S|=k'|S|\geqslant \sum_{v\in S} k_v.\end{aligned}$$

2. $|S|\leqslant k$. We have

$$\begin{aligned}|P_S| &\geqslant (2k-t)|S|-\binom{|S|}{2}\geqslant (2k-t)|S|-\frac{1}{2}|S|^2\\ &\geqslant (2k-t)|S|-\frac{1}{2}k|S|=\left(\frac{3}{2}k-t\right)|S|>k'|S|\geqslant \sum_{v\in S} k_v,\end{aligned}$$

   since $k'<3k/2-t$.

3. $|S|\geqslant n-k$. Let $X=V-S$. Then

$$\begin{aligned}|P_S| &= |E-E(X)|=|E|-|E(X)|\\ &\geqslant m-\binom{|X|}{2}\geqslant m-\frac{1}{2}|X|^2\geqslant \sum_{v\in V} k_v-\frac{1}{2}|X|k\\ &=\sum_{v\in S} k_v+\sum_{v\in X} k_v-\frac{1}{2}|X|k\geqslant \sum_{v\in S} k_v+\frac{k}{2}|X|-\frac{1}{2}|X|k=\sum_{v\in S} k_v. \qquad \square\end{aligned}$$

3. Trees of height at least 5

We consider first trees $T = T_{r,H}$ with $H \geqslant 5$. Let $C = Q(m)$. We will first colour the levels $0, \ldots, H-1$ of T, i.e. all of the non-leaf vertices. Choose C' to be the greatest integer such that $C' \leqslant C$ and C' is relatively prime to r. Then clearly $C' \geqslant \lfloor C/r \rfloor r + 1 > C - r$. We use the colour set $\{1, \ldots, C'\}$. Let $S = \{1, \ldots, C'\}$. Initially we will just use the colours in S; the remaining colours will be used later.

Since we aim to use each pair of colours at most once, it is helpful to use a *colours pairs graph* G, with vertex set S (the colours). Initially $G = K_{C'}$, but it is helpful to think of G evolving as vertices of T are coloured, so that the edge set of G is equal to the pairs of colours not yet used.

If $p \neq C'/2$, we will define the *difference* (*set*) D_p to be the set of colour pairs

$$D_p = \{\{i, (i + p) \bmod C'\} \mid i = 1, \ldots, C'\}$$

Here, and in what follows, we will take the values of $x \bmod y$, where x is an integer and y is a positive integer, to lie between 1 and y. Although this is a set of unordered colour pairs, it is often helpful to think of the elements of D_p as ordered pairs $(i, i + p \,(\mathrm{mod}\, C'))$. Note that $D_p = D_{C'-p}$, so that there are $\lfloor (C'-1)/2 \rfloor$ distinct difference sets.

For each $i = 0, 1, \ldots H-2$, we will choose r difference sets $D(p_{i,1}), \ldots, D(p_{i,r})$ where for some p_i, $p_{i,j} = (p_i + (j-1)r^{H-2-i}) \bmod (C')$, and all the sets $D(p_{i,j})$ are distinct.

It is easy to see that this is possible. For suppose that we have chosen the sets for $i = 0, 1, \ldots, I-1$. Thus we have Ir sets D_p, and since each set $D_p = D_{C'-p}$, we have used sets D_p for $2Ir$ values of p. Each of these values of p forbids r values for p_I, so the total number forbidden is at most $2(H-2)r^2$. Since there are $C'-1$ possible values of p, we can choose the required sets provided that $C' - 1 > 2(H-2)r^2$.

We will use the difference sets so chosen to colour some of the non-leaf part of the tree. Consider the levels $0, 1, \ldots, H-1$ of the tree, each regarded as being ordered from left to right in the obvious way. On level i, starting from the left, we can find $\lfloor r^i / rC' \rfloor$ complete sets of rC' consecutive vertices, i.e. $\lfloor r^i / rC' \rfloor rC'$ vertices in all. We will first colour only the set A_i consisting of the remaining $r^i - \lfloor r^i / rC' \rfloor rC'$ vertices in level i (this will of course be all of the vertices in the first few levels). We also set B_i to be the rightmost $r^i - \lfloor r^i / C' \rfloor C'$ vertices of level i. Note that (i) $0 < |A_i| < rC'$, (ii) $0 < |B_i| < C'$ and (iii) the vertices of A_{i+1} are the children of the vertices of B_i.

We first colour the root with the colour C (the root is the whole of the set A_0). Now suppose that we have coloured set A_i with colours

$$a, a + r^{H-1-i} \,(\mathrm{mod}\, C'), a + 2r^{H-1-i} \,(\mathrm{mod}\, C'), \ldots$$

Then since r^{H-1-i} is relatively prime to C', each colour will recur only after a gap of C' vertices, hence each consecutive block of C' vertices uses the full set of colours $1, \ldots, C'$ once each. Also the vertices of B_i all have distinct colours. We now colour the vertices of A_{i+1} as follows: Since each element of B_i has r children in A_{i+1}, we can partition A_{i+1} in the obvious way into r disjoint sets $P_1, \ldots, P_r$ where for each

$j = 1, 2, \ldots, r$, P_j consists of the jth child of each element of B_i. We will use the r difference sets $D(p_{i,1}), \ldots, D(p_{i,r})$. Thus for a vertex v in B_i coloured x, the jth child of v is coloured $(x + p_{i,j}) \bmod C'$. Since for each $j = 1, 2, \ldots, r-1$,

$$p_{i,j+1} = p_{i,j} + r^{H-2-i} \bmod C',$$

we obtain a colouring of A_{i+1} of the form

$$a', a' + r^{H-2-i} (\bmod C'), a' + 2r^{H-2-i} (\bmod C'), \ldots$$

as required.

It now remains to colour the first $\lfloor r^i/rC' \rfloor rC'$ vertices in levels $0, 1, \ldots, H-1$. Suppose the first level with uncoloured vertices is level I (necessarily $I > 0$). Then these $\lfloor r^I/rC' \rfloor rC'$ uncoloured vertices have $\lfloor r^i/rC' \rfloor C'$ parent vertices in level $I-1$, divided into $\lfloor r^i/rC' \rfloor$ sets each with a full set of colours. To colour level I, we choose such a set, and select the jth child of each for some j. Now choose an unused difference set D_p, and colour the jth child of the vertex v having colour x with colour $(x + p) \bmod C'$. Repeat until all vertices in level I are coloured, Then repeat for level $I+1$, etc. We use a total of $\lfloor r^I/rC' \rfloor r$ difference sets for level I.

We have now coloured all the non-leaf vertices of the tree, using a total of

$$\sum_{i=1}^{H-1} \left(\left\lfloor \frac{r^i}{rC'} \right\rfloor r + r \right) \leqslant \frac{1}{C'} \sum_{i=1}^{H-1} r^i + (H-1)r$$

difference sets. Note that the $\lfloor r^I/rC' \rfloor r$ difference sets used for the second stage at each level can be chosen arbitrarily provided that they have not been used already.

Also each colour occurs on level $H-1$ either $\lfloor r^{H-1}/C' \rfloor$ or $\lceil r^{H-1}/C' \rceil$ times. Now let $z = C - C'$, so that $0 \leqslant z < r$. We now wish to use up all the colour pairs involving one of the colours $C'+1, \ldots, C$. For this we select C' vertices $w_1, \ldots, w_{C'}$ of level $H-1$ with colours $1, \ldots, C'$, respectively, and z further vertices $v_1, \ldots, v_z$ with colours $1, \ldots, z$, respectively. Recolour $v_1, \ldots, v_z$ with colours $C'+1, \ldots, C'+z = C$. Now colour $i-1$ of the children of the vertex v_i with the colours $C'+1, \ldots, C'+i-1$. Then for $i = 1, 2, \ldots, z$ colour the remaining children of v_i with any colours from $1, \ldots, C'$ not already used adjacent to colour i. Finally, for each $i = 1, 2, \ldots, z$, there will be at most C' colours j from $\{1, \ldots, C'\}$ for which the colour pairs $(C'+i, j)$ has not been used; for such pairs, colour one child of w_j with colour $C'+i$. We have now used all the colour pairs involving the colours $C'+1, \ldots, C$.

We must now colour the remaining leaves. Let q_i be the number of uncoloured leaves adjacent to a vertex of colour i, $i = 1, 2, \ldots, C'$. It follows from the above that for each i

$$\left\lfloor \frac{r^{H-1}}{C'} \right\rfloor r - r - z \leqslant q_i \leqslant \left\lceil \frac{r^{H-1}}{C'} \right\rceil r.$$

Recall the colour pairs graph, with vertex set $1, 2, \ldots, C'$ and edge set equal to the set of colour pairs not so far appearing on an edge of the tree.

We can complete the colouring of the tree if (and only if) we can orient the edges of the colour pairs graphs so that for each $i = 1, 2, \ldots, C'$ the outdegree $d_+(i)$ of vertex

i is at least q_i. For then we can colour the q_i leaves adjacent to the vertices of colour i with q_i of the out-neighbours of i in the colour pairs graph.

The number of edges in the colour pairs graph, $|E(G)|$ is certainly at least $\sum_{i=1}^{C'} q_i$. Now the number of difference sets which we have (totally or partially) used is at most

$$\frac{1}{C'}\sum_{i=1}^{H-1} r^i + (H-1)r.$$

Hence the colour pairs graph contains at least

$$\left\lfloor \frac{C'-1}{2} \right\rfloor - \frac{1}{C'}\sum_{i=1}^{H-1} r^i - (H-1)r$$

complete difference sets D_p, with each p satisfying $1 \leqslant p \leqslant \lfloor (C'-1)/2 \rfloor$. All except $(H-1)r$ of these can be chosen with p as small as possible. Let

$$k = \left\lfloor \frac{C'-1}{2} \right\rfloor - \frac{1}{C'}\sum_{i=1}^{H-1} r^i = \left\lfloor \frac{C'-1}{2} \right\rfloor - \frac{1}{C'}\left(\frac{r}{r-1}\right)(r^{H-1}-1).$$

Then the colour pairs graph will contain all except possibly $(H-1)r$ of the difference sets $D_1, \dots, D_k$. Let $t = (H-1)r$, and let $k_{\max} = \lceil r^{H-1}/C' \rceil r$ and $k_{\min} = \lfloor r^{H-1}/C' \rfloor r - r - z$. Then by Lemma 2.3 it will be possible to colour the leaves of the trees provided that

$$k_{\max} \leqslant k - t + \frac{k^2}{4(C'-k)} \quad \text{and} \quad k_{\min} \geqslant k/2.$$

The first inequality is the more difficult and we deal with this first. It is equivalent to

$$k^2 \geqslant 4(C'-k)(k_{\max} - k + t).$$

We first show that this inequality is satisfied provided that $r^H \geqslant 10^{12}$. We do this by bounding the various terms as multiples of C'. First note that

$$C' > (2m)^{1/2} - r \geqslant (2r^H)^{1/2} - r = 2^{1/2} r^{H/2} - r \geqslant r^{H/2} + (2^{1/2}-1) r^{5/2} - r$$

$$\geqslant r^{H/2} + (2^{1/2}-1)2^{3/2} r - r > r^{H/2} \geqslant 10^6.$$

Hence $r/C' < r/r^{H/2} = 1/r^{H/2-1}$. Now $r^{H/2-1} = (r^H)^{(1/2-1/H)} > r^{H/4} \geqslant 10^3$, hence

$$\frac{r}{C'} < \frac{1}{1000} = 0.001. \tag{1}$$

Next we show that $(H-1)r < r^{3H/10}$. For we have $H \geqslant 5$, so that $r^{3H/10} = r^{H/5} r^{H/10} \geqslant r.r^{H/10} > r(10) \geqslant r(H-1)$ provided $H \leqslant 10$. But if $H \geqslant 10$, we have $r^{3H/10} = r^{H/10} r^{H/5} \geqslant r.r^{H/5} > r(100) \geqslant r(H-1)$ provided $H \leqslant 100$. Finally if $H \geqslant 100$,

$$r^{3H/10-1} = \exp(\log_e r(3H/10-1)) \geqslant \frac{1}{2}\left(\frac{3H}{10} - 1\right)^2 (\log_e 2)^2$$

$$\geqslant \frac{1}{2}\left(\frac{3H}{10} - \frac{H}{100}\right)^2 (\log_e 2)^2 = \frac{1}{2}\left(\frac{29H}{100}\right)^2 (\log_e 2)^2$$

$$\geqslant \frac{1}{2}(29)\left(\frac{29H}{100}\right)(\log_e 2)^2 > H.$$

So now

$$\frac{r(H-1)}{C'} < \frac{r^{3H/10}}{r^{H/2}} = \frac{1}{r^{H/5}} < \frac{1}{250} = 0.004. \tag{2}$$

Next note that

$$\left| \left\lfloor \frac{C'-1}{2} \right\rfloor - \frac{C'}{2} \right| \leqslant 1.$$

Recall that $C' \geqslant 10^6$ so that $1 \leqslant 10^{-6}C'$. It follows that

$$\left| \left\lfloor \frac{C'-1}{2} \right\rfloor - \frac{C'}{2} \right| \leqslant 10^{-6}C'. \tag{3}$$

We now consider $|m/C' - C'/2|$. First note that $\frac{1}{2}(C'+r)^2 \geqslant m \geqslant \frac{1}{2}(C'-2)^2$, so that $\frac{1}{2}(r^2 + 2C'r) \geqslant m - \frac{1}{2}C'^2 \geqslant \frac{1}{2}(4 - 4C')$, from which we obtain

$$\frac{1}{2}\left(\frac{r^2}{C'} + 2r\right) \geqslant \frac{m}{C'} - \frac{1}{2}C' \geqslant \frac{1}{2}\left(\frac{4}{C'} - 4\right).$$

Hence

$$\left| \frac{m}{C'} - \frac{C'}{2} \right| \leqslant \frac{C'}{2}\left(\left(\frac{r}{C'}\right)^2 + 2\left(\frac{r}{C'}\right)\right) \leqslant \frac{C'}{2}(10^{-6} + 2.10^{-3}) < 0.0011C'.$$

It follows that

$$\left| \frac{m}{C'}\left(\frac{r-1}{r}\right) - \frac{C'}{2}\left(\frac{r-1}{r}\right) \right| < 0.0011C'. \tag{4}$$

We now estimate

$$\left| k_{\max} - \frac{C'}{2}\left(\frac{r-1}{r}\right) \right|.$$

Recall that $k_{\max} = \lceil r^{H-1}/C' \rceil r$. Hence using (1)we have

$$\left| k_{\max} - \frac{r^H}{C'} \right| \leqslant r \leqslant 0.001C'. \tag{5}$$

Also

$$\left| r^H - \left(\frac{r-1}{r}\right) m \right| = 1.$$

Hence

$$\left| \frac{r^H}{C'} - \left(\frac{r-1}{r}\right) \frac{m}{C'} \right| = \frac{1}{C'} \leqslant 10^{-12}C'. \tag{6}$$

So now, by (4)–(6) we have

$$\begin{aligned} \left| k_{\max} - \frac{C'}{2}\left(\frac{r-1}{r}\right) \right| &\leqslant \left| k_{\max} - \frac{r^H}{C'} \right| + \left| \frac{r^H}{C'} - \left(\frac{r-1}{r}\right)\frac{m}{C'} \right| \\ &\quad + \left| \frac{m}{C'}\left(\frac{r-1}{r}\right) - \frac{C'}{2}\left(\frac{r-1}{r}\right) \right| \\ &\leqslant 10^{-3}C' + 10^{-12}C' + 0.0011C' < 0.002101C'. \end{aligned} \tag{7}$$

Now we similarly estimate

$$\left|k-\frac{C'}{2}\left(\frac{r-1}{r}\right)\right|.$$

Note first that

$$\left|\frac{r}{r-1}(r^{H-1}-1)-\frac{m}{r}\right|=\left|\frac{1}{r-1}((r^H-r)-(r^H-1))\right|=1.$$

Hence

$$\begin{aligned}\left|\frac{1}{C'}\left(\frac{r}{r-1}\right)(r^{H-1}-1)-\frac{C'}{2r}\right| &\leqslant \left|\frac{1}{C'}\left(\frac{r}{r-1}\right)(r^{H-1}-1)-\frac{m}{C'r}\right| \\ &\quad+\left|\frac{m}{C'r}-\frac{C'}{2r}\right|=\frac{1}{C'}+\frac{1}{r}\left|\frac{m}{C'}-\frac{C'}{2}\right| \\ &\leqslant 0.00055C'. \end{aligned} \tag{8}$$

Now by (3) and (8),

$$\begin{aligned}\left|k-\frac{C'}{2}\left(\frac{r-1}{r}\right)\right| &= \left|\left(\left\lfloor\frac{C'-1}{2}\right\rfloor-\frac{1}{C'}\left(\frac{r}{r-1}\right)(r^{H-1}-1)\right)-\frac{C'}{2}\left(\frac{r-1}{r}\right)\right| \\ &= \left|\left(\left\lfloor\frac{C'-1}{2}\right\rfloor-\frac{C'}{2}\right)-\frac{1}{C'}\left(\frac{r}{r-1}\right)(r^{H-1}-1)+\frac{C'}{2r}\right| \\ &\leqslant \left|\left\lfloor\frac{C'-1}{2}\right\rfloor-\frac{C'}{2}\right|+\left|\frac{1}{C'}\left(\frac{r}{r-1}\right)(r^{H-1}-1)-\frac{C'}{2r}\right| \\ &\leqslant 0.000001C'+0.00055C'=0.000551C'. \end{aligned} \tag{9}$$

Now by (7) and (9) we have

$$\begin{aligned}|k_{\max}-k+t| &\leqslant \left|k_{\max}-\frac{C'}{2}\left(\frac{r-1}{r}\right)\right|+\left|k-\frac{C'}{2}\left(\frac{r-1}{r}\right)\right|+|t| \\ &\leqslant 0.002101C'+0.000551C'+0.008C' \\ &= 0.010652C'. \end{aligned}$$

From (9), we obtain

$$k\geqslant\frac{C'}{2}\left(\frac{r-1}{r}\right)-0.0005511C'\geqslant\frac{C'}{4}-0.000551C'=0.249449C'.$$

Thus we have $C'-k\leqslant 0.750551C'$. So finally, $4(C'-k)(k_{\max}-k+t)<0.032C'^2$, while $k^2\geqslant 0.06C'^2$. We now deal with the second inequality $k_{\min}\geqslant k/2$. By a similar calculation to that above, we estimate

$$\left|k_{\min} - \frac{C'}{2}\left(\frac{r-1}{r}\right)\right| \leqslant \left|k_{\min} - \frac{r^H}{C'}\right| + \left|\frac{r^H}{C'} - \left(\frac{r-1}{r}\right)\frac{m}{C'}\right| + \left|\frac{m}{C'}\left(\frac{r-1}{r}\right) - \frac{C'}{2}\left(\frac{r-1}{r}\right)\right|$$

$$\leqslant 2r + t + \frac{1}{C'} + 0.0011C'$$

$$\leqslant 0.002C' + 0.008C' + 10^{-12}C' + 0.0011C' < 0.0112C'.$$

Hence $k_{\min} > (C'/2)((r-1)/r) - 0.0112C'$ while, from (9),

$$k \leqslant \frac{C'}{2}\left(\frac{r-1}{r}\right) + 0.000551C'$$

so

$$k_{\min} - k/2 > \frac{C'}{4}\left(\frac{r-1}{r}\right) - 0.0112C' - 0.0002755C' \geqslant \frac{C'}{8} - 0.0114755C' > 0.$$

as required.

4. Trees of height 3

Unfortunately the construction used in Section 3 fails if the height is 3 or 4, so we need to give special constructions in these cases. We first consider trees of height 3. First note that in this case we have $n = (r^4 - 1)/(r-1)$, and $m = (r/(r-1))(r^3 - 1)$. Let $C = Q(m)$ and $C' = C - 1$. We colour the vertices level by level.

Level 0: We colour the root vertex with colour C.

Level 1: Let $d = (r, C')$, the greatest common divisor of r and C'. We divide the vertices in level 1 into batches of size C'/d (starting from the the left). The last batch will usually be incomplete, and there may only be one batch. We colour the first batch with colours $r, 2r, \ldots, (C'/d)r$, then the second batch with colours $r+1$, $2r+1, \ldots, (C'/d)r+1$, etc., with all colours taken modulo C'.

Level 2: We now choose r differences D_{p_i} with $p_i = \lfloor (C'-1)/2 \rfloor - i + 1$, $i = 1, \ldots, r$. If a vertex in level 1 has colour x, we colour its j child with colour $x + p_j \pmod{C'}$. The result of this is that each batch of C' vertices in level 2, starting from the left, uses all the colours $1, \ldots, C'$. Thus each of these colours occurs in level 2 either $\lfloor r^2/C' \rfloor$ or $\lceil r^2/C' \rceil$ times.

We now have to use up the remaining colour pairs involving colour C. To do this we choose one vertex in level 2, for each of the necessary colours, and colour one of its children with colour C.

We must now colour the remaining vertices in level 3 (the leaves). Let q_i be the number of uncoloured leaves adjacent to a vertex of colour i, $i = 1, 2, \ldots, C'$. It follows from the above that for each i $\lfloor r^2/C' \rfloor r - 1 \leqslant q_i \leqslant \lceil r^2/C' \rceil r$. Also, the number of differences D_p which we have (totally or partially) used is exactly r, all chosen as desired. Let $k=\lfloor (C'-1)/2\rfloor - r$. Then the colour pairs graph contains all the differences $D_1,\ldots,D_k$. Let $k_{\max}=\lfloor r^2/C'\rfloor r$ and $k_{\min}=\lfloor r^2/C'\rfloor r-1$. Then by Lemma 2.3 it will be possible to colour the leaves of the trees provided that $k^2 \geqslant 4(C'-k)(k_{\max}-k)$ and $k_{\min}\geqslant k/2$. We will show that these inequalities are satisfied if $r\geqslant 239$. First note that $C=Q(m)\geqslant\sqrt{2m}\geqslant\sqrt{2r^3}$. Hence we have

$$r/C\leqslant 1/\sqrt{2r} \tag{10}$$

and

$$1/C\leqslant 1/\sqrt{2r^3}. \tag{11}$$

First we estimate $|k-(C/2)((r-1)/r)|$. We have

$$\left|k-\frac{C}{2}\right| \leqslant \left|\left\lfloor\frac{C-2}{2}\right\rfloor-\frac{C}{2}\right|+\left|k-\frac{C-2}{2}\right|\leqslant\frac{3}{2}+r. \tag{12}$$

Also

$$\left|\frac{C}{2}-\frac{C}{2}\left(\frac{r-1}{r}\right)\right|=\frac{C}{2r}.$$

Hence

$$\left|k-\frac{C}{2}\left(\frac{r-1}{r}\right)\right|\leqslant\frac{3}{2}+r+\frac{C}{2r}. \tag{13}$$

Now we estimate $|k_{\max}-(C/2)((r-1)/r)|$. Recall that $k_{\max}=\lceil r^2/C'\rceil r$. Hence we have

$$\left|k_{\max}-\frac{r^3}{C'}\right|\leqslant r. \tag{14}$$

Also

$$\left|r^3-\left(\frac{r-1}{r}\right)m\right|=1.$$

Hence

$$\left|\frac{r^3}{C'}-\left(\frac{r-1}{r}\right)\frac{m}{C'}\right|=\frac{1}{C'}. \tag{15}$$

Finally, we have

$$\left|m-\frac{C(C-1)}{2}\right|\leqslant C-1,$$

so that

$$\left|\frac{m}{C-1}-\frac{C}{2}\right|\leqslant 1,$$

and so

$$\left|\frac{m}{C-1}\left(\frac{r-1}{r}\right)-\frac{C}{2}\left(\frac{r-1}{r}\right)\right|\leqslant 1. \tag{16}$$

So now we have, by (14)–(16),

$$\begin{aligned}\left|k_{\max}-\frac{C}{2}\left(\frac{r-1}{r}\right)\right| &\leqslant \left|k_{\max}-\frac{r^3}{C'}\right|+\left|\frac{r^3}{C'}-\left(\frac{r-1}{r}\right)\frac{m}{C'}\right|\\ &\quad+\left|\frac{m}{C'}\left(\frac{r-1}{r}\right)-\frac{C}{2}\left(\frac{r-1}{r}\right)\right|\\ &\leqslant r+\frac{1}{C'}+1\\ &\leqslant r+2. \end{aligned}\tag{17}$$

Thus by (17) and (13), and using the estimates (10) and (11), we obtain

$$\begin{aligned}|k_{\max}-k| &\leqslant \left|k_{\max}-\frac{C}{2}\left(\frac{r-1}{r}\right)\right|+\left|k-\frac{C}{2}\left(\frac{r-1}{r}\right)\right|\\ &\leqslant \frac{3}{2}+r+\frac{C}{2r}+r+2\leqslant\frac{7}{2}+2r+\frac{C}{2r}\\ &= C\left(\frac{7}{2C}+\frac{2r}{C}+\frac{1}{2r}\right)\leqslant C\left(\frac{7}{2\sqrt{2r^3}}+\frac{2}{\sqrt{2r}}+\frac{1}{2r}\right)\\ &\leqslant C\times 0.09424.\end{aligned}$$

Also by (12),

$$k\geqslant\frac{C}{2}-\frac{3}{2}-r=C\left(\frac{1}{2}-\frac{3}{2C}-\frac{r}{C}\right)\geqslant C\left(\frac{1}{2}-\frac{3}{2\sqrt{2r^3}}-\frac{1}{\sqrt{2r}}\right)\geqslant C\times 0.45397$$

Hence $C'-k=C-1-k\leqslant C\times 0.5459$. So we have finally that

$$4(C'-k)(k_{\max}-k)\times 0.09424\times 0.5459\times C^2<0.20579C^2$$

while $k^2\geqslant(C\times 0.45397)^2>0.206C^2$ so the first inequality is satisfied. For the second, note that

$$\left|k_{\min}-\frac{C}{2}\left(\frac{r-1}{r}\right)\right|\leqslant r+3$$

and

$$\left|k-\frac{C}{2}\left(\frac{r-1}{r}\right)\right|\leqslant\frac{3}{2}+r+\frac{C}{2r}.$$

Hence

$$k_{\min}\geqslant\frac{C}{2}\left(\frac{r-1}{r}\right)-r-3$$

and

$$k\leqslant\frac{C}{2}\left(\frac{r-1}{r}\right)+\frac{3}{2}+r+\frac{C}{2r}.$$

So

$$\begin{aligned}
k_{\min} - k/2 &\geqslant \frac{C}{2}\left(\frac{r-1}{r}\right) - r - 3 - \frac{C}{4}\left(\frac{r-1}{r}\right) - \frac{3}{4} - \frac{r}{2} - \frac{C}{4r} \\
&\geqslant \frac{C}{8} - \frac{15}{4} - \frac{3r}{2} - \frac{C}{4r} \\
&\geqslant C\left(\frac{1}{8} - \frac{15}{4C} - \frac{3r}{2C} - \frac{1}{4r}\right) \\
&\geqslant C\left(\frac{1}{8} - \frac{15}{4\sqrt{2r^3}} - \frac{3}{2\sqrt{2r}} - \frac{1}{4r}\right) \\
&\geqslant 0.054C > 0.
\end{aligned}$$

Hence both the necessary inequalities are satisfied.

5. Trees of height 4

We must now consider trees of height 4, for which we need another special construction. First note that in this case we have

$$n = (r^5 - 1)/(r - 1),$$

and

$$m = \frac{r}{r-1}(r^4 - 1).$$

Let $C = Q(m)$. We choose C' as large as possible subject to the following conditions: (i) C' is odd; (ii) C' is relatively prime to r; (iii) $C' \leqslant C - 2$. It is not hard to see that $C' > C - r$. For if $\lfloor (C-1)/r \rfloor r + 1$ is odd then it satisfies the conditions unless it greater than $C - 2$, in which case we could use $\lfloor (C-1)/r \rfloor r - 1$ instead, which is greater than $C - r$ provided $r \geqslant 4$. Otherwise if $\lfloor (C-1)/r \rfloor r + 1$ is even, then r must be odd, so r is relatively prime to $\lfloor (C-1)/r \rfloor r + 2$ and to $\lfloor (C-1)/r \rfloor r - 2$. Provided $r \geqslant 6$, one of these satisfies the conditions and is greater than $C - r$.

As before, we colour the vertices level by level.

Level 0: We colour the root vertex with colour C.

Level 1: There are r vertices in level 1, say $u_0, \ldots, u_{r-1}$. Colour u_i with colour $ir^2 + 1 \pmod{C'}$.

Level 2: There are r^2 vertices in level 2, say $v_0, \ldots, v_{r^2-1}$. Colour v_i with colour $ir + 2 \pmod{C'}$.

Note that the children of the node in level 1 which is coloured $ir^2 + 1$ have colours $ir^2 + jr + 2$ for $j = 0, 1, \ldots, r$, hence the differences D_k used are for $k = jr + 1$, $j = 0, 1, \ldots, r - 1$.

Level 3: There are r^3 vertices in level 3, say $w_0, \ldots, w_{r^3-1}$. Colour w_i with colour $i + 3 \pmod{C'}$.

Note that the children of the node in level 2 which is coloured $ir+2$ have colours $ir+j+3$ for $j=0,1,\dots,r$, hence the differences D_k used are for $k=j+1, j=0,1,\dots,r-1$.

These differences are separate from the ones used at the previous level apart from D_1. Hence the only possibility of a colour pair having been repeated occurs if a colour in level 2 is the same as a colour in level 1. This occurs if

$$ir^2 + 1 = i'r + 2 \pmod{C'}$$

for some i, i'. However if s is such that $rs = 1 \pmod{C'}$, then we have

$$i' = ir - s \pmod{C'}$$

hence for each value of i there is just one value of i' which causes a repeated colour. Thus there are at most r vertices in level 2 with the same colour as a vertex in level 1. One child of each of these (the child with difference 1) will need to be recoloured; we do this with colour $C-1$.

We now have to use up the remaining colour pairs involving the colours $C'+1,\dots,C$. To do this we recolour $C-C'-1$ of the vertices in level 3 (which have distinct colours) with the colours $C'+1,\dots,C-1$. We can then colour some of the leaves attached to these vertices to use up all colour pairs which involve only the colours $C'+1,\dots,C$. We then colour the remaining leaves adjacent to any vertices of these extra colours with suitable colours, and then use up any remaining colour pairs involving the extra colours by colouring up to r of the leaves adjacent to each of the regular colours $1,\dots,C'$.

Let q_i be the number of uncoloured leaves adjacent to a vertex of colour $i, i=1,2,\dots,C'$. It follows from the above that for each i

$$\left\lfloor \frac{r^3}{C'} \right\rfloor r - 2r \leqslant q_i \leqslant \left\lceil \frac{r^3}{C'} \right\rceil r.$$

The number of difference sets D_k used is $2r-1$. Let $k = \lfloor (C'-1)/2 \rfloor$. Then the colour pairs graph contains all except $2r-1$ of the differences $D_1,\dots,D_k$. Hence letting $t = 4r-2, k_{\max} = \lceil r^3/C' \rceil r$ and $k_{\min} = \lfloor r^3/C' \rfloor r - 2r$, we know from Lemma 2.3 that we can complete the harmonious colouring of the tree if

$$k^2 \geqslant 4(C'-k)(k_{\max} - k + t) \quad \text{and} \quad k_{\min} \geqslant k/2.$$

For the first inequality we have

$$C = Q(m) \geqslant \sqrt{2m} \geqslant \sqrt{2}r^2$$

so that $C' \geqslant \sqrt{2}r^2 - r$. It follows that

$$\frac{r}{C'} \leqslant \frac{1}{\sqrt{2}r - 1}$$

and

$$\frac{1}{C'} \leqslant \frac{1}{\sqrt{2}r^2 - r}$$

Now we have

$$\left|k-\frac{C'}{2}\right| = \left|\left\lfloor\frac{C'-1}{2}\right\rfloor - \frac{C'}{2}\right| \leqslant 1.$$

Also

$$\left|\frac{C'}{2}-\frac{C'}{2}\left(\frac{r-1}{r}\right)\right| = \frac{C'}{2r}.$$

Hence

$$\left|k-\frac{C'}{2}\left(\frac{r-1}{r}\right)\right| \leqslant 1+\frac{C'}{2r}.$$

Also $k \geqslant C'/2-1$, so

$$C'-k \leqslant C'/2+1.$$

Now we estimate $|k_{\max}-(C'/2)((r-1)/r)$. As in Section 3, we have

$$\begin{aligned}\left|k_{\max}-\frac{C'}{2}\left(\frac{r-1}{r}\right)\right| &\leqslant \left|k_{\max}-\frac{r^4}{C'}\right| \\ &\quad + \left|\frac{r^4}{C'}-\left(\frac{r-1}{r}\right)\frac{m}{C'}\right| + \left|\frac{m}{C'}\left(\frac{r-1}{r}\right)-\frac{C'}{2}\left(\frac{r-1}{r}\right)\right| \\ &\leqslant r+\frac{1}{C'}+\frac{1}{2}\left(\frac{r^2}{C'}+2r\right).\end{aligned}$$

Thus we can now calculate

$$\begin{aligned}|k_{\max}-k+t &\leqslant \left|k_{\max}-\frac{C'}{2}\left(\frac{r-1}{r}\right)\right| + \left|k-\frac{C'}{2}\left(\frac{r-1}{r}\right)\right| + |t| \\ &\leqslant 1+\frac{C'}{2r}+r+\frac{1}{C'}+\frac{1}{2}\left(\frac{r^2}{C'}+2r\right)+4r-2 \\ &\leqslant 6r+\frac{C'}{2r}+\frac{1}{C'}+\frac{r^2}{2C'} \\ &= C'\left(\frac{6r}{C'}+\frac{1}{2r}+\left(\frac{1}{C'}\right)^2+\frac{1}{2}\left(\frac{r}{C'}\right)^2\right).\end{aligned}$$

Now if $r \geqslant 39$, we have $C'-k \leqslant 0.5005C'$ and $|k_{\max}-k+t| \leqslant 0.1238C'$, so that $|4(C'-k)(k'-k+t)| \leqslant 0.2479(C')^2$ while $k^2 \geqslant 0.249(C')^2$. Hence the first inequality holds. For the second, we have

$$\left|k_{\min}-\frac{C'}{2}\left(\frac{r-1}{r}\right)\right| \leqslant 3r+\frac{1}{C'}+\frac{1}{2}\left(\frac{r^2}{C'}+2r\right)$$

while

$$\left|k-\frac{C'}{2}\left(\frac{r-1}{r}\right)\right| \leqslant 1+\frac{C'}{2r}.$$

So we have

$$k_{\min} \geqslant \frac{C'}{2}\left(\frac{r-1}{r}\right) - 3r - \frac{1}{C'} - \frac{1}{2}\left(\frac{r^2}{C'} + 2r\right)$$

and

$$k \leqslant \frac{C'}{2}\left(\frac{r-1}{r}\right) + 1 + \frac{C'}{2r}.$$

It follows that

$$\begin{aligned}
k_{\min} - k/2 &\geqslant \frac{C'}{4}\left(\frac{r-1}{r}\right) - 3r - \frac{1}{C'} - \frac{1}{2}\left(\frac{r^2}{C'} + 2r\right) - \frac{1}{2} - \frac{C'}{4r} \\
&= \frac{C'}{4}\left(\frac{r-1}{r}\right) - 4r - \frac{1}{C'} - \frac{r^2}{2C'} - \frac{1}{2} - \frac{C'}{4r} \\
&= C'\left(\frac{1}{8} - \frac{4r}{C'} - \left(\frac{1}{C'}\right)^2 - \frac{1}{2}\left(\frac{r}{C'}\right)^2 - \frac{1}{2C'} - \frac{1}{4r}\right) \\
&\geqslant 0.044C' > 0
\end{aligned}$$

as required.

6. Remaining cases

We have shown that $h(T_{r,H}) = Q(m)$ for trees $T_{r,H}$, in the following cases:

1. $H \geqslant 5$, $r^H \geqslant 10^{12}$;
2. $H = 4$, $r \geqslant 39$;
3. $H = 3$, $r \geqslant 239$.

This was done by showing that certain inequalities, whose terms depend on r and H, hold in the above cases and guarantee the existence of the appropriate colouring.

This leaves a finite number of cases unresolved. In some of these cases the above inequalities do in fact hold. This can be shown by calculating the terms precisely rather than estimating them. All outstanding cases with more than 10^7 vertices can be dealt with in this way.

Then we are left with a residue of about 200 cases which must be tackled separately. There seems to be little hope of producing a neat description of the colouring of these smaller trees, so we must resort to computer calculation. Fortunately, because all the remaining cases have fewer than 10^7 vertices, it is possible to calculate explicit colourings for each of them.

The method used to find colourings for these smaller trees is roughly as follows: We calculate $Q(m)$ and aim to use the colours $1, \dots, Q(m)$. For $H \geqslant 5$, the levels $0, 1, \dots, H-1$ are coloured randomly (subject to the requirement that the colouring be harmonious) level by level. We then use the technique of orienting the edges of the colour pairs graph to complete the colouring — in this case we must actually find a suitable orientation (whose existence will not in general be guaranteed by Lemma 2.3). An efficient algorithm for finding such an orientation (or proving its non-existence) was given in [3]. We used this method with some refinements to speed it up. In some cases several attempts at the random part of the colouring were needed before a colouring which could be extended to the whole tree was found.

For $H = 3, 4$, constructions similar to those described in Sections 4 and 5 were used to colour the non-leaf vertices, the extension to the leaves is as above. For a few of the smallest trees, some ad hoc variations were necessary to find a colouring.

Fortunately, having found an explicit colouring, it can be checked by a simple computer procedure that it is indeed a valid harmonious colouring with $Q(m)$ colours. This was done in all the cases where an explicit colouring was found.

The only case with $H \geqslant 3$ for which a $Q(m)$ colouring cannot be found is $T_{2,3}$. This is easily seen as follows. $T_{2,3}$ has 15 vertices and 14 edges, hence $Q(m) = 6$. Levels 1 and 2 have 6 vertices altogether. If two of these vertices have the same colour c, then they cannot have a common neighbour, and since they each have degree three, they have 6 neighbours altogether. All of these neighbours must have distinct colours, different from c, hence $h \geqslant 7$. On the other hand, if all 6 vertices in levels 1 and 2 have distinct colours, then the root vertex cannot use any of these colours, so again $h \geqslant 7$. A 7-colouring is easily found so $h(T_{2,3}) = 7$.

We can now state:

Theorem 6.1. *Let $T_{r,H}$ be the complete r-ary tree of height H. Then*
(i) $h(T_{r,1}) = r + 1$,
(ii) $h(T_{r,2}) = \lceil 3(r + 1)/2 \rceil$,
(iii) $h(T_{2,3}) = 7$,
(iv) $h(T_{r,H}) = Q(m)$ otherwise.

Proof. Case (i) is trivial, case (ii) is due to Mitchem [8], the remaining cases are dealt with in this paper. □

7. Conclusion

The exact determination of $h(G)$ for specific graphs G seems to be quite difficult, even for highly structured graphs. Some other exact results are mentioned in [2]. However some very simple graphs remain unsolved, e.g. square grids or graphs of maximum degree 2, although there are partial results in both of these cases.

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